

THE CONJUGACY CLASSES OF FINITE NONSOLVABLE SUBGROUPS IN THE PLANE CREMONA GROUP.

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ABSTRACT. The aim of this paper is to give a finer geometric description of the algebraic varieties parametrizing conjugacy classes of nonsolvable subgroups in the plane Cremona group.

Keywords: Cremona group, del Pezzo surface, conic bundle, automorphisms group.
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1. INTRODUCTION.

The classification of finite subgroups in the plane Cremona group over the field \mathbb{C} denoted by $\text{Cr}_2(\mathbb{C})$ is a classical problem. The history of this problem begins with the work of E. Bertini [2], where are classified the conjugacy classes of subgroups of order 2 in $\text{Cr}_2(\mathbb{C})$. There were obtained three families of conjugacy classes now called as involution de Jonquières, Geiser and Bertini. However, the classification was incomplete, and the proof was not rigorous. Only recently in [1] was obtained complete and short proof.

In 1895 Kantor [11] and Wiman [14] gave a description of finite subgroups in $\text{Cr}_2(\mathbb{C})$. The list was fairly comprehensive, but was not full in the following aspects. Firstly, for a given finite subgroup on this list could not be defined, whether it is contained in the Cremona group or not. Secondly, the question of conjugacy between the subgroups was not considered.

Modern approach to the problem was initiated by the work of Manin [12] and continued in works of Iskovskikh [7], [8], [9]. In the paper [12] is established a clear link between the classification of conjugacy classes of finite subgroups of the Cremona group and the classification of G -minimal rational surfaces (S, G) and G -equivariant birational maps between them. The consideration is divided into two cases: when S is a del Pezzo surface, and when S is a conic bundle.

Definition 1.1. Let G be a finite group. A G -surface is a triple (S, G, ρ) , where S is a nonsingular projective surface, and ρ is a monomorphism of the group G to the automorphisms group of the surface S . For brevity, G -surface will be denoted by (S, G) .

Let G be a finite subgroup in $\text{Cr}_2(\mathbb{C})$ with an embedding $\theta : G \hookrightarrow \text{Cr}_2(\mathbb{C})$. It turns out that the action of G on \mathbb{P}^2 can be *regularized*, i.e there exists a smooth rational surface S and a birational map $\mu : S \dashrightarrow \mathbb{P}^2$ such that $\mu^{-1} \circ \theta(G) \circ \mu$ is a subgroup of automorphisms of S .

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Certainly, any regularization is not unique. For example, if we blow up any G -orbit of points on S . Two distinct G -surfaces (S, G) and (S', G) define two conjugate embeddings $\theta : G \rightarrow \mathrm{Cr}_2(\mathbb{C})$ and $\theta' : G \rightarrow \mathrm{Cr}_2(\mathbb{C})$ respectively, iff there exist a G -equivariant birational map $\zeta : S \dashrightarrow S'$.

Definition 1.2. A G -surface (S, G) is called G -minimal, if any G -equivariant birational morphism $S \rightarrow Y$ onto a smooth G -surface Y is a G -isomorphism.

Theorem 1.3 ([12]). *There are two types of the rational G -minimal surfaces (S, G) :*

- S is a del Pezzo surface, and $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$;
- S has a G -equivariant structure of conic bundle $\phi : S \rightarrow \mathbb{P}^1$, and $\mathrm{Pic}(S)^G \simeq \mathbb{Z}^2$.

Notation 1.4. *The classes of G -minimal rational surfaces from the first and the second cases of the Theorem 1.3 will be denoted respectively as \mathbb{D} and \mathbb{CB} .*

More recently, I.V. Dolgachev and V.A. Iskovskih [4] improved the list of Kantor and Wiman. The answer was obtained in terms of action of the groups G on the del Pezzo surfaces and on the conic bundles. It was considered question about conjugacy of the finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$, using the theory of elementary links of V.A. Iskovskih (see [10]). For general case this paper is currently the most precise classification of conjugacy classes of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$. I note that J. Blanc in [3] obtained a more precise classification in case of finite cyclic subgroups.

However in [4] explicit equations of G -minimal surfaces (S, G) in weighted projective spaces and explicit descriptions of actions of G on surfaces S were obtained only in case of Del Pezzo surfaces. Also description of groups G , acting on G -minimal conic bundles (S, G, ϕ) , was given only in terms of groups extensions. In other words, for a given abstract finite group it is still impossible, using [4], to say whether the group is isomorphic to a subgroup of $\mathrm{Cr}_2(\mathbb{C})$. Also classification of conjugacy classes of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ has some gaps. If $G \subset \mathrm{Cr}_2(\mathbb{C})$ is regularized as a subgroup of automorphisms of a conic bundle $\phi : S \rightarrow \mathbb{P}^1$ with $K_S^2 = 1$, or 2, and $(S, G, \phi) \in \mathbb{CB}$. I will show it consistently for $K_S^2 = 1$ and 2.

Let $K_S^2 = 1$. Consider [4, Section 8.1, Pages 534-535]. There is stated non-existence of triples $(S, G, \phi) \in \mathbb{CB}$ with $K_S^2 = 1$ and ample divisor $-K_S$. However, it's wrong. An example of such triples is presented in [13, Section 6.2.3, Theorem 6.8]. In this case the authors of [4] applied an old incorrect version of [4, Theorem 5.7]. This version existed until J. Blanc reported about a mistake to I. Dolgachev. I note that in published version of [4] Theorem 5.7 is presented in correct form. Unfortunately, for large volume of work, the authors forgot to update some conclusions from the theorem.

Let $K_S^2 = 2$. In [4, Section 8.1, Pages 535] is stated: if $(S, G, \phi) \in \mathbb{CB}$ with $K_S^2 = 2$ and non-ample divisor $-K_S$ then the surface S is *exceptional conic bundle* (see Definition 3.3). In other words the surface S contains two smooth non-intersecting rational (-3) -curves. This is also wrong. In [13, Section 5.1.1, Theorem 5.4] is presented an example of triple $(S, G, \phi) \in \mathbb{CB}$ with $K_S^2 = 2$ and nef, non-ample divisor $-K_S$.

In the paper [13] I continue classification of G -minimal conic bundles, which was begun in [4]. For given arbitrary value of K_S^2 , it was constructed a method of classification by means of explicit equations of G -minimal conic bundles (S, G) in weighted projective spaces and explicit descriptions of the actions of G on the Picard

group $\text{Pic}(S)$ and on the surface S . The classification is carried on completely for $K_S^2 > 0$. If $K_S^2 \leq 0$ then the G -minimal conic bundle (S, G) is birationally rigid. So there is no question about conjugacy (see [4, Section 8]).

The aim of this paper is to give a finer geometric description of the algebraic varieties parameterizing conjugacy classes of finite nonsolvable subgroups in $\text{Cr}_2(\mathbb{C})$, applying methods of papers [4] and [13]. It is obtained explicit equations of G -minimal surfaces (S, G) in weighted projective spaces and explicit descriptions of actions of G on surfaces S . Also all possibilities for the groups G are fully described.

In [4, Section 9] were stated the following problems for $\text{Cr}_2(\mathbb{C})$:

- Find the finer classification of the conjugacy classes of de Jonquières groups.
- Give a finer geometric description of the algebraic varieties parameterizing conjugacy classes.

This article gives a solution of these problems for the nonsolvable finite subgroups in $\text{Cr}_2(\mathbb{C})$.

It's important to note that investigation method described in the paper can be employed to solve these problems for all finite subgroups in $\text{Cr}_2(\mathbb{C})$, i.e. not necessary nonsolvable. However due to large amount of routine work investigation was conducted only for nonsolvable subgroups.

The paper has the following structure. In Section 2 I study surfaces from the class \mathbb{D} , i.e. the G -minimal del Pezzo surfaces (S, G) , where $\text{Pic}(S)^G \simeq \mathbb{Z}$ and G is a finite nonsolvable group. I will apply here results of [4]. There are no my results in this section.

In Section 3 I study surfaces from the class \mathbb{CB} , i.e. the G -minimal surfaces (S, G, ϕ) , where a morphism $\phi : S \rightarrow \mathbb{P}^1$ defines a G -equivariant conic bundle structure, $\text{Pic}(S)^G \simeq \mathbb{Z}^2$, and G is a finite nonsolvable group. The main my results are described in this section.

In Section 4 I study conjugacy classes of embeddings $G \rightarrow \text{Cr}_2(\mathbb{C})$ defined by G -minimal surfaces (S, G) , for all finite nonsolvable subgroups $G \subset \text{Cr}_2(\mathbb{C})$. Here I reprove results in [4, Section 7] for the sake of completeness.

In Section 5 I present a list of the finite nonsolvable subgroups in $\text{Cr}_2(\mathbb{C})$, obtained on the basis of results of sections 2 and 3.

This work is dedicated to my supervisor Vasily Alekseevich Iskovskikh, who initiated my study of the Cremona group. I am very grateful to Yuri Gennadievich Prokhorov and Ilya Alexandrovich Tyomkin for useful tips and remarks.

The base field is assumed everywhere to be \mathbb{C} . Throughout this paper we will use the following notations.

- ε_n denotes a primitive n -th root of unity.
- S_n denotes the permutation group of degree n .
- A_n denotes the alternating group of degree n .
- Consider a subgroup $A_5 \subset \text{PGL}(2, \mathbb{C})$, which is isomorphic to the icosahedral automorphisms group, and the standard projection $\psi : \text{SL}(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C})$. Then \bar{A}_5 denotes the group $\psi^{-1}(A_5)$, which is isomorphic to the binary icosahedral group.
- $A.B$, where A and B are some abstract groups, is one of the possible extensions with help of the exact sequence: $0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0$.
- Let H be an abstract group. Then $H \wr S_n$ will denote the semidirect product $H^n \rtimes S_n$, where S_n is the symmetric group, acting on H^n by permuting the factors.

- $A \triangle_D B$ is a diagonal product of abstract groups A and B over their common homomorphic image D (i.e. the subgroup of $A \times B$ of pairs (a, b) , such that $\alpha(a) = \beta(b)$ for some epimorphisms $\alpha : A \rightarrow D$, $\beta : B \rightarrow D$).
- $\mathbb{P}(a_1, \dots, a_n)$, where $a_i \in \mathbb{Z}$, $i = 1, \dots, n$, is the weighted projective space, with the set of weights (a_1, \dots, a_n) .

2. CASE OF DEL PEZZO SURFACES.

In this section we study the surfaces $(S, G) \in \mathbb{D}$, i.e. S is a G -minimal del Pezzo surface, and $\text{Pic}(S)^G \simeq \mathbb{Z}$. The groups G are supposed to be finite nonsolvable. We will apply here results of [4]. There are no author's results in this section.

Recall that a surface S is called a del Pezzo surface, if S is smooth, and $-K_S$ is ample. It's well known that $1 \leq K_S^2 \leq 9$. We will carry our investigation, considering different values of K_S^2 .

In the next theorem we study the case $K_S^2 = 9$. In this case $S \simeq \mathbb{P}^2$.

Theorem 2.1. *Let $(S, G) \in \mathbb{D}$, $K_S^2 = 9$, and G be a finite nonsolvable group. Then $S \simeq \mathbb{P}^2$ with the coordinates $(x_0 : x_1 : x_2)$, and G is any finite nonsolvable subgroup of $\text{Aut}(\mathbb{P}^3) \simeq \text{PGL}(3, \mathbb{C})$. The subgroup $G \subset \text{PGL}(3, \mathbb{C})$ can be conjugated to one of the following subgroups.*

- (1) H is a group, consisting of maps

$$(x_0 : x_1 : x_2) \mapsto (ax_0 + bx_1 : cx_0 + dx_1 : x_2).$$

The image of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$$

in $\text{PGL}(2, \mathbb{C})$ under the natural projection $GL(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C})$ is isomorphic to A_5 . The group H is isomorphic to $\mathbb{Z}_n \times \bar{A}_5$, $n \geq 1$.

- (2) The icosahedral group A_5 isomorphic to $L_2(5)$. It leaves invariant a non-singular conic $C \subset \mathbb{P}^2$.
- (3) The Klein group isomorphic to $L_2(7)$. This group is realized as automorphism group of the Klein's quartic $x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = 0$.
- (4) The Valentiner group isomorphic to A_6 . It can be realized as automorphism group of the nonsingular plane sextic

$$10x_0^3x_1^3 + 9x_2x_0^5 + x_1^6 - 45x_0^2x_1^2x_2^2 - 135x_0x_1x_2^4 + 27x_2^6 = 0.$$

Proof. The statement follows directly from [4, Corollary 4.6, Theorems 4.7, 4.8]. We need only to check the isomorphism $H \simeq \mathbb{Z}_n \times \bar{A}_5$, $n \geq 1$ in the first case of theorem. It follows from [4, Lemma 4.5, case (i)]. \square

In the next theorem we consider the case $K_S^2 = 8$.

Theorem 2.2. *Let $(S, G) \in \mathbb{D}$, $K_S^2 = 8$, and G be a finite nonsolvable group. Then $S \simeq \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with the coordinates $(x_0 : x_1, t_0 : t_1)$. We will employ definition of the group $\text{St}(A_5)$ (see Notation 3.5), and define involution*

$$\tau : (x_0 : x_1, t_0 : t_1) \mapsto (t_0 : t_1, x_0 : x_1).$$

We have the following possibilities for G .

- (1) The subgroup $G \subset \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is conjugate to the subgroup $\text{St}(A_5) \wr \langle \tau \rangle$.

- (2) The subgroup $G \subset \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is conjugate to the subgroup $H \times \langle \tau \rangle$, where H is the image of the diagonal embedding of $St(A_5)$ in $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$.

Proof. One knows that if S is a del Pezzo surface with $K_S^2 = 8$ then $S \simeq \mathbb{F}_0$ or \mathbb{F}_1 . However in the second case the exceptional section of ruled surface \mathbb{F}_1 is G -invariant. Therefore the pair (\mathbb{F}_1, G) is not G -minimal. Hence $S \simeq \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

It's well known that $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq PGL(2, \mathbb{C}) \wr \langle \tau \rangle$. Whence G is generated by a nonsolvable subgroup $H \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ and by an element $\eta = \mu \circ \tau$, where $\mu \in PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. We write $\mu = (B, B')$, where $B, B' \in PGL(2, \mathbb{C})$. For any $\varsigma = (A, A') \in PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ we have

$$(2.1) \quad \eta \circ \varsigma \circ \eta^{-1} = (BA'B^{-1}, B'AB'^{-1}).$$

Let's study the structure of group H , applying Goursat's Lemma (see [4, Lemma 4.1]). Consider projections $\pi_i : PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C}) \rightarrow PGL(2, \mathbb{C})$, $i = 1, 2$ on the first and the second factor respectively. We get $H \simeq \pi_1(H) \triangle_D \pi_2(H)$, where $D \simeq \text{Im}(\pi_1|H) / \text{Ker}(\pi_2|H)$. Obviously, either $\text{Im}(\pi_1|H) \simeq A_5$ or $\text{Im}(\pi_2|H) \simeq A_5$ (see Klein's classification of finite nonsolvable subgroups in $PGL(2, \mathbb{C})$ in [4, Section 5.5]). The group A_5 is simple. Therefore $D \simeq 1$ or A_5 .

Suppose that $D \simeq 1$. From (2.1) we get $\text{Im}(\pi_1|H) \simeq \text{Im}(\pi_2|H) \simeq A_5$. Hence H can be conjugated to $St(A_5) \times St(A_5)$. We will prove that $B, B' \in St(A_5)$. Suppose that it doesn't hold. Then from (2.1) we get $St(A_5) \rtimes B, St(A_5) \rtimes B' \subset PGL(2, \mathbb{C})$. But $St(A_5)$ is a maximal finite subgroup of $PGL(2, \mathbb{C})$. Contradiction. We get the first case of the theorem.

Suppose that $D \simeq A_5$. Then H is conjugated to the image of diagonal embedding of $St(A_5)$ in $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. Arguing as above, we get $B, B' \in St(A_5)$. By (2.1) these elements define the same inner automorphism of A_5 . Hence $B = B'$. We get the second case of the theorem. \square

In the next theorem we consider cases: $K_S^2 = 7$, $K_S^2 = 6$, $K_S^2 = 4$, and $K_S^2 = 1$.

Theorem 2.3. *There are no surfaces $(S, G) \in \mathbb{D}$, such that K_S^2 is equal to either 7, or 6, or 4, or 1, and G is a finite nonsolvable group.*

Proof. Let's consider the case $K_S^2 = 7$. The surface S is presented as a blowing up of two different points in \mathbb{P}^2 . However the strict transform of line, containing this two points, is a G -invariant rational (-1) -curve. Hence the surface S is not G -minimal.

The case $K_S^2 = 6$ follows directly from [4, Theorem 6.3, Corollary 4.6, Theorem 4.7].

The case $K_S^2 = 4$ follows directly from [4, Theorem 6.9].

The case $K_S^2 = 1$ follows directly from [4, Table 8]. \square

In the next theorem we consider the case $K_S^2 = 5$.

Theorem 2.4. *Let $(S, G) \in \mathbb{D}$, $K_S^2 = 5$, and G be a finite nonsolvable group. Introduce on \mathbb{P}^2 the coordinates $(T_0 : T_1 : T_2)$. Then the surface S is isomorphic to the blowing up of \mathbb{P}^2 at points: $(0 : 0 : 1)$, $(0 : 1 : 0)$, $(1 : 0 : 0)$ and $(1 : 1 : 1)$. The group $\text{Aut}(S)$ is isomorphic to S_5 , and is generated by the maps:*

$$(2.2) \quad \begin{aligned} (T_0 : T_1 : T_2) &\mapsto (T_1 : T_2 : T_0), \\ (T_0 : T_1 : T_2) &\mapsto (T_2 : -T_0 + T_2 : -T_1 + T_2), \\ (T_0 : T_1 : T_2) &\mapsto (T_0(T_2 - T_1) : T_2(T_0 - T_1) : T_0T_2). \end{aligned}$$

The subgroup $G \subset \text{Aut}(S)$ is isomorphic to A_5 or S_5 .

Proof. This follows directly from [4, Theorem 6.4] and arguments of [5, Theorem 8.4.15]. \square

In the next theorem we consider the case $K_S^2 = 3$.

Theorem 2.5. *Let $(S, G) \in \mathbb{D}$, $K_S^2 = 3$, and G be a finite nonsolvable group. Then the surface S can be represented by the following equations in \mathbb{P}^3 with the coordinates $(T_0 : T_1 : T_2 : T_3)$:*

$$T_0^2 T_1 + T_1^2 T_2 + T_2^2 T_3 + T_3^2 T_0 = 0.$$

The group G is isomorphic to S_5 and is generated by the following maps:

$$(2.3) \quad \begin{aligned} (T_0 : T_1 : T_2 : T_3) &\mapsto (T_0 : \varepsilon_5^4 T_1 : \varepsilon_5 T_2 : \varepsilon_5^2 T_3), \\ (T_0 : T_1 : T_2 : T_3) &\mapsto (T_1 : T_2 : T_3 : T_0). \end{aligned}$$

Proof. This follows directly from [4, Theorem 6.14]. \square

In the next theorem we consider the case $K_S^2 = 2$.

Theorem 2.6. *Let $(S, G) \in \mathbb{D}$, $K_S^2 = 2$, and G be a finite nonsolvable group. Then the surface S can be represented by the following equation in $\mathbb{P}(2, 1, 1, 1)$ with the coordinates $(T_0 : T_1 : T_2 : T_3)$:*

$$(2.4) \quad T_3^2 + T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0 = 0.$$

The group $\text{Aut}(S)$ is isomorphic to $\mathbb{Z}_2 \times L_2(7)$. The subgroup $G \subset \text{Aut}(S)$ is isomorphic to either $L_2(7)$, or $\mathbb{Z}_2 \times L_2(7)$.

Proof. This follows directly from [4, Theorem 6.17]. \square

3. CASE OF CONIC BUNDLES

In this section we study the surfaces (S, G, ϕ) in the class \mathbb{CB} , i.e. G -minimal surfaces S with $\text{Pic}(S)^G \simeq \mathbb{Z}^2$, having a G -equivariant conic bundle structure $\phi : S \rightarrow \mathbb{P}^1$. The groups G are supposed to be finite nonsolvable.

Recall (see [4, Item 3.7]) that a rational G -surface (S, G) has a conic bundle structure, if there exist a G -equivariant morphism $\phi : S \rightarrow \mathbb{P}^1$, whose each fiber $F_t = \phi^{-1}(t)$, $t \in \mathbb{P}^1$ is either a nondegenerate plane conic (isomorphic to \mathbb{P}^1) or a reducible reduced conic, i.e. a pair of intersecting lines.

A conic bundle (S, G, ϕ) is said to be *relatively G -minimal*, if the fibres of ϕ do not contain G -orbits, consisting of nonintersecting rational (-1) -curves (i.e. components of reducible fibres — equivalently to $\text{Pic}(S)^G = \phi^* \text{Pic}(\mathbb{P}^1) \oplus \mathbb{Z} \simeq \mathbb{Z}^2$). Recall that a G -surface (S, G) is said to be *G -minimal*, if any G -equivariant birational morphism $S \rightarrow Y$ onto a smooth G -surface Y is a G -isomorphism. It is clear that a G -minimal surface, having a conic bundle structure, is relatively minimal. The inverse statement is not always valid.

Denote by r the number of the reducible fibers of a conic bundle (S, G, ϕ) . By Noether formula we have $r = 8 - K_S^2$, so $K_S^2 \leq 8$. If $K_S^2 = 8$, then S is isomorphic to Hirzebruch's surface \mathbb{F}_n , $n \geq 0$.

Theorem 3.1. *Let $(S, G, \phi) \in \mathbb{CB}$ with $K_S^2 = 8$, and G be a finite nonsolvable group. Then the surface S is isomorphic to Hirzebruch's surface \mathbb{F}_n , $n \geq 0$. The morphism $\phi : S \rightarrow \mathbb{P}^1$ coincides with the standard projection $\mathbb{F}_n \rightarrow \mathbb{P}^1$.*

- (1) Let $n = 0$. Then $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The group $\text{Aut}(\mathbb{F}_0)$ is isomorphic to $PGL(2, \mathbb{C}) \wr S_2$. The subgroup $G \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C}) \subset \text{Aut}(\mathbb{F}_0)$ is isomorphic to one of the following: $A_5 \times B$, $B \times A_5$, $A_5 \triangle_{A_5} A_5$, where B is any finite subgroup of $PGL(2, \mathbb{C})$.
- (2) Let $n > 0$. Then $n > 1$. Consider $\mathbb{F}_n \rightarrow \mathbb{P}(n, 1, 1)$ the blowdown of exceptional section of \mathbb{F}_n . We have

$$\text{Aut}(\mathbb{F}_n) \simeq \mathbb{C}^{n+1} \rtimes (GL(2, \mathbb{C})/\mu_n),$$

where $GL(2, \mathbb{C})/\mu_n$ acts on \mathbb{C}^{n+1} by means of its natural linear representation in the space of binary forms with degree n . The subgroup $G \subset \text{Aut}(\mathbb{F}_n)$ is isomorphic to one of the following groups:

$$(3.1) \quad G \simeq \begin{cases} \mathbb{Z}_m \times A_5, m \geq 1, & \text{if } n \text{ is even;} \\ \mathbb{Z}_m \times \bar{A}_5, m \geq 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $S \simeq \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then G is a nonsolvable subgroup in $\text{Aut}(\mathbb{F}_0) \simeq PGL(2, \mathbb{C}) \wr S_2$. Note that $G \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$, so as $\text{Pic}(S)^G \simeq \mathbb{Z}^2$. We apply Goursat's lemma (see [4, Lemma 4.1]) and Klein's classification of the finite subgroups in $PGL(2, \mathbb{C})$ (see [4, Section 5.5]). We get that $G \simeq B \triangle_D C$, where one of groups B and C is isomorphic to A_5 . Since the group A_5 is simple, the group D is isomorphic to either 1 or A_5 . Therefore the group G is isomorphic to one of the following groups: $A_5 \times B$, $B \times A_5$, $A_5 \triangle_{A_5} A_5$, where B is any finite subgroup of $PGL(2, \mathbb{C})$. Remark that a group $A_5 \triangle_{A_5} A_5$ is conjugate to image of a diagonal embedding of group A_5 to $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$.

Consider the case $S \simeq \mathbb{F}_n$, $n > 0$. Let $\mathbb{F}_n \rightarrow \mathbb{P}(n, 1, 1)$ be the blowdown of the exceptional section of \mathbb{F}_n .

We note that if $n = 1$ then $\mathbb{P}(1, 1, 1)$ is a smooth surface. Hence the triple (S, G, ϕ) is not minimal. Therefore $n \neq 1$.

Introduce the coordinates $(x : t_0 : t_1)$ on $\mathbb{P}(n, 1, 1)$. It's well known (see [4, Theorem 4.10]) that $\text{Aut}(\mathbb{F}_n) \simeq \mathbb{C}^{n+1} \rtimes (GL(2, \mathbb{C})/\mu_n)$. The group \mathbb{C}^{n+1} is generated by maps $(x : t_0 : t_1) \mapsto (x + f_n(t_0, t_1) : t_0 : t_1)$, where f_n is a binary form with degree n . The group $GL(2, \mathbb{C})/\mu_n$ is generated by invertible maps $(x : t_0 : t_1) \mapsto (x : at_0 + bt_1 : ct_0 + dt_1)$. Moreover, we have

$$GL(2, \mathbb{C})/\mu_n \simeq \begin{cases} \mathbb{C}^* \rtimes SL(2, \mathbb{C}), & \text{if } n \text{ is odd;} \\ \mathbb{C}^* \rtimes PGL(2, \mathbb{C}), & \text{if } n \text{ is even.} \end{cases}$$

Consider the sequence of homomorphisms $G \xrightarrow{h_1} GL(2, \mathbb{C})/\mu_n \xrightarrow{h_2} PGL(2, \mathbb{C})$, where h_2 is natural projection. Obviously that the homomorphism h_1 is injective, and $\text{Im}(h_2) \simeq A_5$ (see [4, Section 5.5]).

Thus the group G is isomorphic to a central extension $\mathbb{Z}_m.A_5$, if n is even, or $\mathbb{Z}_m.\bar{A}_5$, if n is odd. Applying [4, Lemma 4.4], we get (3.1). \square

There are no relatively G -minimal conic bundles (S, G, ϕ) , if $K_S^2 = 7$. So there are no G -minimal conic bundles too. If $K_S^2 = 3, 5$ or 6 , then there exists a G -equivariant morphism $(S, G) \rightarrow (S', G)$, where $(S', G) \in \mathbb{D}$ (see [8, Proposition 2.1, Theorem 4.1] and, for example, [13, Section 2]). Thus study of this cases reduces to study of G -minimal del Pezzo surfaces. For other values $K_S^2 = 4, 2, 1, \dots$ relatively G -minimal conic bundles are always G -minimal.

The morphism $\phi : S \rightarrow \mathbb{P}^1$ induces a homomorphism $\phi_* : G \rightarrow \text{Aut}(\mathbb{P}^1)$. We have the following exact sequence

$$(3.2) \quad 1 \rightarrow G_K \rightarrow G \rightarrow G_B \rightarrow 1,$$

where $G_K \simeq \text{Ker}(\phi_*)$, and $G_B \simeq \text{Im}(\phi_*)$.

Also consider the natural representation of group G in the automorphisms group of lattice $\text{Pic}(S)$. By G_0 we denote the kernel of this representation. The group G_0 fixes the divisor classes of the sections with negative self-intersection. Such sections obviously exist. Hence G_0 fixes it pointwisely. Considering one of these sections as a point on a general fibre, we conclude that G_0 is a cyclic group.

Theorem 3.2 ([4, Proposition 5.5]). *Let $(S, G, \phi) \in \mathbb{CB}$ with $K_S^2 \leq 4$, $K_S^2 \neq 3$. Suppose that $G_0 \neq \{1\}$. Then the surface S has an exceptional conic bundle structure (see below).*

Definition 3.3. Define the *exceptional conic bundles*. This is the minimal resolution of singularities of surface, given by the equation in weighted projective space $\mathbb{P}(1, 1, g+1, g+1)$, where $g \geq 1$:

$$Y_g : F_{2g+2}(t_0, t_1) + t_2 t_3 = 0,$$

where F_{2g+2} is a binary form without multiple factors with degree $2g+2$.

After the resolution of indeterminacy points, the projection onto \mathbb{P}^1 with coordinates (t_0, t_1) will induce a conic bundle structure $\phi : \widetilde{Y}_g \rightarrow \mathbb{P}^1$. This conic bundle has reducible fibres over the points from \mathbb{P}^1 , where $F_{2g+2}(t_0, t_1) = 0$. The surface \widetilde{Y}_g contains 2 nonintersecting rational $(-g-1)$ -curves defined by the equations $t_2 = 0$ and $t_3 = 0$. Automorphisms of the surface \widetilde{Y}_g are induced by automorphisms of $\mathbb{P}(1, 1, g+1, g+1)$.

The case of exceptional conic bundles will be considered in Section 3.1.

There is a theorem about the structure of minimal finite groups, acting on the non-exceptional conic bundles.

Theorem 3.4 ([4, Theorem 5.7]). *Let $(S, G, \phi) \in \mathbb{CB}$ with $K_S^2 \leq 4$, $K_S^2 \neq 3$, and Σ be the set of reducible fibres of ϕ . Suppose that $G_0 \simeq 1$. Then one of the following cases occurs.*

- (1) *Case $G_K \simeq \mathbb{Z}_2$. The central involution ι , generating the group G_K , fixes pointwise a smooth bisection C of the fibration ϕ and switches the components of fibres in a subset $\Sigma' \subset \Sigma$. The morphism ϕ defines the linear system g_2^1 on the curve C having branch points in the singular points of the fibres in the set Σ' . Genus of the curve C is equal to $g = (m-2)/2$, where $m = |\Sigma'|$. The group G_B is isomorphic to a subgroup of the automorphism group of curve C modulo the involution defined by g_2^1 .*
- (2) *Case $G_K \simeq \mathbb{Z}_2^2$. Each nontrivial element ι_i , $i = 0, 1, 2$ of the group G_K fixes pointwise an irreducible smooth bisection C_i . The set Σ is partitioned into three subsets $\Sigma_0, \Sigma_1, \Sigma_2$, such that $\Sigma_i = (\Sigma_j \cup \Sigma_k) \setminus (\Sigma_j \cap \Sigma_k)$, $i \neq j \neq k \neq i$. The morphisms $\phi|_{C_i}$, $i = 0, 1, 2$ are branched over the singular points of fibres in subsets Σ_i . The group G_B leaves invariant the set of points $\phi(\Sigma) \in \mathbb{P}^1$ and its partition into three subsets $\phi(\Sigma_i)$, $i = 0, 1, 2$.*

Consider cases of Theorem 3.4 separately. We note that the subgroup $G_B \subset \text{Aut}(\mathbb{P}^1) \simeq \text{PGL}(2, \mathbb{C})$ (see (3.2)) is nonsolvable, since G_K is solvable by Theorem

3.4. Hence $G_B \simeq A_5$ (see Klein's classification of the finite subgroups in $PGL(2, \mathbb{C})$ in [4, Section 5.5]). We will use the fact in sections 3.2, 3.3, 3.4 without mentioning.

- (1) Case $G_K \simeq \mathbb{Z}_2$, and $\Sigma' = \Sigma$. This case will be investigated in Section 3.2.
- (2) Case $G_K \simeq \mathbb{Z}_2$, and $\Sigma' \neq \Sigma$. This case will be investigated in Section 3.3.
- (3) Case $G_K \simeq \mathbb{Z}_2^2$. This case will be investigated in Section 3.4.

We will often use the following facts about finite nonsolvable subgroups $\bar{P} \subset SL(2, \mathbb{C})$ (see [4, Section 5.5]). Obviously that $\bar{P} \simeq \bar{A}_5$. Any group of this type is conjugated to a group with the following generators:

$$(3.3) \quad g_1 = \begin{pmatrix} \varepsilon_{10} & 0 \\ 0 & \varepsilon_{10}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} \varepsilon_5 - \varepsilon_5^4 & \varepsilon_5^2 - \varepsilon_5^3 \\ \varepsilon_5^2 - \varepsilon_5^3 & -\varepsilon_5 + \varepsilon_5^4 \end{pmatrix}.$$

Notation 3.5. We will denote a group generated by (3.3) as $St(\bar{A}_5)$. Its image in $PGL(2, \mathbb{C})$ we will denote as $St(A_5)$.

Consider the natural representation of $St(\bar{A}_5)$ in space of polynomials $\mathbb{C}[t_0, t_1]$. Space of relative invariants of the representation is generated by the following Gröndformens:

$$(3.4) \quad \begin{aligned} \Phi_1 &= T_0^{30} + T_1^{30} + 522(T_0^{25}T_1^5 - T_0^5T_1^{25}) - 10005(T_0^{20}T_1^{10} + T_0^{10}T_1^{20}), \\ \Phi_2 &= -(T_0^{20} + T_1^{20}) + 228(T_0^{15}T_1^5 - T_0^5T_1^{15}) - 494T_0^{10}T_1^{10}, \\ \Phi_3 &= T_0T_1(T_0^{10} + 11T_0^5T_1^5 - T_1^{10}). \end{aligned}$$

Since $\bar{A}_5/(\pm 1) \cong A_5$ is a simple group and all Gröndformens have even degree, we easily see that $g(\Phi_i) = \Phi_i$, $i = 1, 2, 3$, for any $g \in St(\bar{A}_5)$. In other words, the characters are trivial.

Notation 3.6. We will denote space of invariants of group \bar{A}_5 generated by this Gröndformens as $\mathcal{I}^{St(\bar{A}_5)}$.

3.1. Case of exceptional conic bundles. In this section we will prove the following theorem.

Theorem 3.7. Let $(S, G, \phi) \in \mathbb{CB}$ be an exceptional conic bundle, and G be a finite nonsolvable group. Then the surface S can be represented as the minimal resolution of singularities of surface given by the equation in the weighted projective space $\mathbb{P}(1, 1, g+1, g+1)$, $g \geq 1$ with coordinates $(t_0 : t_1 : t_2 : t_3)$:

$$Y_g : F_{2g+2}(t_0, t_1) + t_2t_3 = 0,$$

where $F_{2g+2} \in \mathcal{I}^{St(\bar{A}_5)}$ is a binary form without multiple factors with degree $2g+2$. The morphism $\phi : S \rightarrow \mathbb{P}^1$ is induced by the map $\phi' : Y_g \dashrightarrow \mathbb{P}^1$ given by

$$\phi' : (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1).$$

The group G is isomorphic to

$$G \simeq \begin{cases} D_n \times \bar{A}_5, & n \geq 1, \text{ if } g \text{ is even;} \\ D_n \times A_5, & n \geq 1, \text{ if } g \text{ is odd.} \end{cases}$$

All possibilities for G occur. The group G is generated by the maps:

$$\begin{aligned}
(3.5) \quad & (t_0 : t_1 : t_2 : t_3) \mapsto (\varepsilon_{10} t_0 : \varepsilon_{10}^{-1} t_1 : t_2 : t_3), \\
& (t_0 : t_1 : t_2 : t_3) \mapsto (it_1 : it_0 : t_2 : t_3), \\
& (t_0 : t_1 : t_2 : t_3) \mapsto ((\varepsilon_5 - \varepsilon_5^4) t_0 + (\varepsilon_5^2 - \varepsilon_5^3) t_1 : (\varepsilon_5^2 - \varepsilon_5^3) t_0 + (-\varepsilon_5 + \varepsilon_5^4) t_1 : t_2 : t_3), \\
& (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : \varepsilon_m t_2 : \varepsilon_m^{-1} t_3), \\
& (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : t_3 : t_2),
\end{aligned}$$

where $m = n$, if g is odd, and $m = 2n$, otherwise.

Proof. By [4, Proposition 5.3] we have $\text{Aut } Y_g \simeq N.P$, where $N \simeq \mathbb{C}^* \rtimes \mathbb{Z}_2$ is a group generated by the maps:

$$\begin{aligned}
(3.6) \quad & (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : t_3 : t_2), \\
& (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : ct_2 : c^{-1} t_3), \quad c \in \mathbb{C}, \quad c \neq 0.
\end{aligned}$$

And P is the subgroup of $PGL(2, \mathbb{C})$, leaving the form $F_{2g+2}(t_0, t_1)$ semi-invariant. Obviously that $P \simeq A_5$. We conjugate the subgroup $P \subset PGL(2, \mathbb{C})$ to the subgroup $St(A_5)$. Then $F_{2g+2} \in \mathcal{I}^{St(A_5)}$. Whence we get that the group $\text{Aut}(Y_g)$ is generated by maps (3.5) and (3.6). Hence

$$\text{Aut}(Y_g) \simeq \begin{cases} (N/\mu_2) \times \bar{A}_5, & \text{if } g \text{ is even;} \\ N \times A_5, & \text{if } g \text{ is odd,} \end{cases}$$

where the group μ_2 acts by $(t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : -t_2 : -t_3)$. It follows from the description of exceptional conic bundles (see [4, Section 5.2]) that the triple (S, G, ϕ) is minimal, iff the group G permutes points: $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. Therefore $G \cap N \simeq D_n$. Further arguments are obvious. \square

3.2. Case, when $G_0 \simeq 1$, $G_K \simeq \mathbb{Z}_2$, and $\Sigma' = \Sigma$. Here we will apply arguments of [13, Section 3.1].

The group G_K is generated by involution ι . By [13, Theorem 3.2] we get $S/\iota \simeq \mathbb{F}_e$. The morphism $\pi : S \rightarrow S/\iota \simeq \mathbb{F}_e$ is G -equivariant, and a faithful action of the group G_B is defined on \mathbb{F}_e (see exact sequence (3.2)). Recall (see Theorem 3.4) that the morphism $\pi : S \rightarrow \mathbb{F}_e$ is branched over a nonsingular hyperelliptic curve C . Let $\bar{C} = \pi(C)$. We consider cases $e = 0$ and $e > 0$ in Theorems 3.8 and 3.10 respectively.

We make some preparations before statement of Theorem 3.8. Introduce the coordinates $(x_0 : x_1, t_0 : t_1)$ on $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The morphism $\phi : S \rightarrow \mathbb{P}^1$ induces projection $(x_0 : x_1, t_0 : t_1) \mapsto (t_0 : t_1)$. The curve \bar{C} is represented by the equation:

$$(3.7) \quad \text{Equat}(\bar{C}) = p_0(t_0, t_1)x_0^2 + 2p_1(t_0, t_1)x_0x_1 + p_2(t_0, t_1)x_1^2 = 0,$$

where p_i , $i = 0, 1, 2$ are binary forms with degree $2d$. Note that the degree is even, so as the divisor class $\bar{C} \in 2\text{Pic}(\mathbb{F}_0)$ (where $2\text{Pic}(\mathbb{F}_0) \subset \text{Pic}(\mathbb{F}_0) \simeq \mathbb{Z}^2$ is the even sublattice). The form $\text{Disc}(\bar{C}) = p_0p_2 - p_1^2$ has no multiple factors, since \bar{C} is nonsingular. We will apply the Segre embedding $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ to represent the surface S by equations. Introduce the coordinates $(x : y : z : w)$ on \mathbb{P}^3 . This embedding is given by $\nu : (x_0 : x_1, t_0 : t_1) \mapsto (x_0t_0 : x_0t_1 : x_1t_0 : x_1t_1)$. We choose some polynomials $F_i(x, y, z, w)$, $i = 0, \dots, 2d - 2$, such that $x_0^i x_1^{2d-2-i} \text{Equat}(\bar{C}) = \nu^*(F_i)$.

The surface S is represented by the equations in $\mathbb{P}(d^d, 1^4)$ with the coordinates $u_i, x, y, z, w, i = 0, \dots, d-1$:

$$(3.8) \quad \begin{aligned} u_i u_j &= F_{i+j}, \quad 0 \leq i \leq j \leq d-1, \\ x^{j-i} u_i &= u_j z^{j-i}, \quad y^{j-i} u_i = u_j w^{j-i}, \quad 0 \leq i < j \leq d-1, \\ xw &= yz. \end{aligned}$$

Theorem 3.8. *Let $(S, G, \phi) \in \mathbb{CB}$, and G be a finite nonsolvable group. Suppose that $G_0 \simeq 1$, $G_K \simeq \langle \iota \rangle \simeq \mathbb{Z}_2$, $\Sigma' = \Sigma$, and $S/\iota \simeq \mathbb{F}_0$. Then the surface S is represented by equations (3.8). The morphism $\phi : S \rightarrow \mathbb{P}^1$ is given by*

$$\phi : (u_0 : \dots : u_{d-1} : x : y : z : w) \mapsto \begin{cases} (x : y), & \text{if } (x : y) \neq (0 : 0); \\ (z : w), & \text{if } (z : w) \neq (0 : 0). \end{cases}$$

There is defined a faithful action of G_B (see exact sequence (3.2)) on \mathbb{F}_0 , and $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. One of the following cases occurs.

- (1) The subgroup $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ is conjugated to the subgroup $1 \times St(A_5)$, and $G \simeq \mathbb{Z}_2 \times A_5$.
- (2) The subgroup $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ is conjugated to the diagonal embedding $St(A_5) \hookrightarrow PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$, and

$$G \simeq \begin{cases} \bar{A}_5, & \text{if } d \text{ is even in (3.7),} \\ \mathbb{Z}_2 \times A_5, & \text{otherwise.} \end{cases}$$

All cases exist. And all possibilities for G occur.

In all cases G acts on S by the following way. Embedding $G_B \simeq St(A_5)$ to $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ defines a unique embedding $St(\bar{A}_5) \hookrightarrow SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. This defines an action of $St(\bar{A}_5)$ on the surface S given by equation (3.8). The action of $St(\bar{A}_5)$ on coordinates $u_i, i = 0, \dots, d-1$ coincides with the action on monomials $x_0^i x_1^{d-1-i}, i = 0, \dots, d-1$, respectively. An action of G is generated by the action of $St(\bar{A}_5)$ and by the map

$$(u_0 : \dots : u_{d-1} : x : y : z : w) \rightarrow (-u_0 : \dots : -u_{d-1} : x : y : z : w).$$

Proof. Recall that $G_B \simeq A_5$. The subgroup $G_B \subset \text{Aut}(\mathbb{F}_0)$ acts trivially on $\text{Pic}(\mathbb{F}_0)$. Hence $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. We apply Goursat's Lemma (see [4, Lemma 4.1]) to study the subgroups $A_5 \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. Consider projections $\pi_i : PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C}) \rightarrow PGL(2, \mathbb{C}), i = 1, 2$ on the first and the second factor respectively. We get $G_B \simeq \pi_1(G_B) \triangle_D \pi_2(G_B)$, where $D \simeq \text{Im}(\pi_1|_{G_B}) / \text{Ker}(\pi_2|_{G_B})$. The group A_5 is simple. Therefore $D \simeq 1$ or A_5 . These cases corresponds respectively to cases 1 and 2 of the theorem.

It's need to check existence of the nonsingular curve $\bar{C} \subset \mathbb{F}_0$ for each of these cases. In the first case this curve obviously exists. Because we can choose binary forms $p_i \in \mathcal{I}^{St(\bar{A}_5)}, i = 0, 1, 2$ in (3.7) without multiple and common factors (see the generators of $\mathcal{I}^{St(\bar{A}_5)}$ in (3.4)). It remains to verify existence of the nonsingular curve \bar{C} in the second case. Also we need to show that the parameter d in (3.7) can be odd and even. This follows from the next lemma.

Lemma 3.9. *There exist nonsingular curves $\bar{C} \in \mathbb{F}_0$ with odd and even parameter d given by equation (3.7) and invariant under the diagonal action of group $St(A_5)$ on $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Consider the linear space of polynomials $\mathbb{C}[x, y]$. The space has the natural structure of $SL(2, \mathbb{C})$ -module. Denote by $R_n \subset \mathbb{C}[x, y]$ the subspace of polynomials with degree n . We have $\text{Equat}(\bar{C}) \in R_2 \otimes R_{2d}$. It's known (see [6, Exercise 11.11]) that $R_2 \otimes R_{2d} \simeq R_{2d+2} \oplus R_{2d} \oplus R_{2d-2}$ as $SL(2, \mathbb{C})$ -module.

Consider a linear system \mathcal{J} of $St(\bar{A}_5)$ -invariant curves with bidegree $(2, 2d)$ in \mathbb{F}_0 . Obviously, we have $\mathcal{J} \simeq (R_2 \otimes R_{2d})^{St(\bar{A}_5)} \simeq R_{2d+2}^{St(\bar{A}_5)} \oplus R_{2d}^{St(\bar{A}_5)} \oplus R_{2d-2}^{St(\bar{A}_5)}$.

First, we prove existence of a nonsingular curve \bar{C} with odd parameter d . We take $d = 30k + 15, k \in \mathbb{N}, k \geq 2$. It's easy to check (see (3.4)) that each set $R_{2d+2}^{St(\bar{A}_5)}, R_{2d}^{St(\bar{A}_5)}$ and $R_{2d-2}^{St(\bar{A}_5)}$ is not empty. To apply Bertini theorem, we need to study the base points of system \mathcal{J} .

We have $(x_0 t_1 - x_1 t_0)^2 R_{2d-2}(t_0, t_1)^{St(\bar{A}_5)} \in \mathcal{J}$. It's easy to check that $R_{2d-2}^{St(\bar{A}_5)} = \Phi_3^4 R_{60k-20}^{St(\bar{A}_5)}$, and $\Phi_2^2 R_{60(k-1)}^{St(\bar{A}_5)} \subset R_{60k-20}^{St(\bar{A}_5)}$ (see (3.4)). The space $R_{60(k-1)}^{St(\bar{A}_5)}$ has not a common factor, since $k \geq 2$. Hence the base points of \mathcal{J} lie in the union of sets: $x_0 t_1 - x_1 t_0 = 0, \Phi_2(t_0, t_1) = 0$ and $\Phi_3(t_0, t_1) = 0$.

Consider the projection $\xi : R_2 \otimes R_{2d} \simeq R_{2d+2} \oplus R_{2d} \oplus R_{2d-2} \rightarrow R_{2d+2}$. It is given by the polynomial $p_0 t_0^2 + 2p_1 t_0 t_1 + p_2 t_1^2$. This polynomial defines intersection of the curve \bar{C} and of diagonal $x_0 t_1 - x_1 t_0 = 0$. We have $\Phi_2 \Phi_3 R_{60k}^{St(\bar{A}_5)} \subset R_{2d+2}^{St(\bar{A}_5)}$. The space $R_{60k}^{St(\bar{A}_5)}$ has not a common factor, since $k \geq 2$. Hence we can take a polynomial $\text{Equat}(\bar{C}) \in R_2 \otimes R_{2d}$, such that $\xi(\text{Equat}(\bar{C})) = \Phi_2 \Phi_3 h(t_0, t_1)$, where the forms $\Phi_2, \Phi_3, h(t_0, t_1)$ have not pairwise common factors. Therefore the base points of \mathcal{J} lie in the union of sets: $\Phi_2(t_0, t_1) = 0$ and $\Phi_3(t_0, t_1) = 0$. However by choose of $\text{Equat}(\bar{C})$ we get that the curve \bar{C} is nonsingular at these sets.

It remains to prove existence of a nonsingular curve \bar{C} with even parameter d . We take $d = 30k, k \in \mathbb{N}, k \geq 3$. It is easy to check (see (3.4)) that each set $R_{2d+2}^{St(\bar{A}_5)}, R_{2d}^{St(\bar{A}_5)}$ and $R_{2d-2}^{St(\bar{A}_5)}$ is not empty. To apply Bertini theorem, we need to study the base points of system \mathcal{J} .

We have $\Phi_1 \mathcal{J}' \subset \mathcal{J}$, where \mathcal{J}' is a linear system of $St(\bar{A}_5)$ -invariant curves with bidegree $(2, 2d - 30)$ in \mathbb{F}_0 . By previous arguments, we know that the base points of \mathcal{J}' lie in the union of sets: $\Phi_2(t_0, t_1) = 0$ and $\Phi_3(t_0, t_1) = 0$. Hence the base points of \mathcal{J} lie in the union of sets: $\Phi_1(t_0, t_1) = 0, \Phi_2(t_0, t_1) = 0$ and $\Phi_3(t_0, t_1) = 0$.

Again consider projection ξ . We have $\Phi_1 \Phi_2 \Phi_3 R_{60(k-1)}^{St(\bar{A}_5)} \subset R_{2d+2}^{St(\bar{A}_5)}$. The space $R_{60(k-1)}^{St(\bar{A}_5)}$ has not a common factor, since $k \geq 3$. Hence we can take a polynomial $\text{Equat}(\bar{C}) \in R_2 \otimes R_{2d}$, such that $\xi(\text{Equat}(\bar{C})) = \Phi_1 \Phi_2 \Phi_3 h(t_0, t_1)$, where the forms $\Phi_1, \Phi_2, \Phi_3, h(t_0, t_1)$ have not pairwise common factors. Again the conditions of Bertini theorem are satisfied. \square

It remains to describe the action of group G . The embedding of the group $G_B \simeq St(A_5)$ to $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ defines a unique embedding $St(\bar{A}_5) \hookrightarrow SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. We note that $\text{Equat}(\bar{C})$ is invariant under the action of $St(\bar{A}_5)$. Hence there is defined an action of $St(\bar{A}_5)$ on the surface S given by equation (3.8). The action of $St(\bar{A}_5)$ on coordinates $u_i, i = 0, \dots, d-1$ coincides with the action on monomials $x_0^i x_1^{d-1-i}, i = 0, \dots, d-1$, respectively. The remaining arguments are obvious. \square

The case $e > 0$ will be considered in the next theorem.

Theorem 3.10. *Let $(S, G, \phi) \in \mathbb{CB}$, and G be a finite nonsolvable group. Suppose that $G_0 \simeq 1$, $G_K \simeq \langle \iota \rangle \simeq \mathbb{Z}_2$, $\Sigma' = \Sigma$, and $S/\iota \simeq \mathbb{F}_n$, $n > 0$. Then there is a G -invariant curve E , which is the preimage of exceptional section \mathbb{F}_e . Consider the contraction of this curve $(S, G, \phi) \rightarrow (S', G, \phi')$, where a map $\phi' : S' \dashrightarrow \mathbb{P}^1$ is defined by ϕ . The surface S' is given by the following equation in $\mathbb{P}(d+e, e, 1, 1)$ with the coordinates $(u : x : t_0 : t_1)$:*

$$(3.9) \quad u^2 + p_0(t_0, t_1)x^2 + 2p_1(t_0, t_1)x + p_2(t_0, t_1) = 0,$$

where p_i , $i = 0, 1, 2$ are binary forms with degree $2d$, $2d+e$, $2d+2e$, respectively. The binary form $p_0p_2 - p_1^2$ has no multiple factors. The map ϕ' is given by

$$\phi' : (u : x : t_0 : t_1) \mapsto (t_0 : t_1).$$

Moreover, e is even. The group G is generated by the maps:

$$\begin{aligned} u &\mapsto -u, \\ (u : x : t_0 : t_1) &\mapsto (u : x + F_e(t_0, t_1) : at_0 + bt_1 : ct_0 + d't_1), \end{aligned}$$

where

$$(3.10) \quad \begin{pmatrix} a & b \\ c & d' \end{pmatrix} \in St(\bar{A}_5),$$

and $F_e(t_0, t_1)$ is a some binary form with degree e , unique for each matrix (3.10). The group G is isomorphic to

$$(3.11) \quad G \simeq \begin{cases} \bar{A}_5, & \text{if } d \text{ is odd,} \\ \mathbb{Z}_2 \times A_5, & \text{if } d \text{ is even.} \end{cases}$$

All possibilities for G occur.

Proof. We will use the following construction to represent the surface S by equations. Consider the morphism $\mathbb{F}_e \rightarrow \mathbb{P}(e, 1, 1)$, which is the blowing down of exceptional section \mathbb{F}_e . Introduce the coordinates $(x : t_0 : t_1)$ on $\mathbb{P}(e, 1, 1)$. The morphism ϕ induces projection $(x : t_0 : t_1) \mapsto (t_0 : t_1)$. The curve \bar{C} will be represented by the following equation in $\mathbb{P}(e, 1, 1)$:

$$p_0(t_0, t_1)x^2 + 2p_1(t_0, t_1)x + p_2(t_0, t_1) = 0.$$

The form $\text{Disc}(\bar{C}) = p_0p_2 - p_1^2$ has no multiple factors, since \bar{C} is nonsingular. We construct a double cover of $\mathbb{P}(e, 1, 1)$, branched along \bar{C} . We get the surface S' given by the equations (3.9).

Note that $\deg(p_0)$ is even, since $\bar{C} \in 2\text{Pic}(\mathbb{F}_e)$. Denote the degree as $2d$. The automorphism group of $\mathbb{P}(e, 1, 1)$ consists of the maps

$$(x : t_0 : t_1) \mapsto (a'x + P_e(t_0, t_1) : b't_0 + c't_1 : d't_0 + v't_1).$$

where P_e is a binary form with degree e . We can choose coefficients b' , c' , d' , v' , so that

$$\begin{pmatrix} b' & c' \\ d' & v' \end{pmatrix} \in SL(2, \mathbb{C}).$$

We have $G_B \simeq A_5 \not\subset \bar{A}_5$ (see (3.2)). Therefore e is even. We conjugate G_B to a group consisting of the following maps

$$(x : t_0 : t_1) \mapsto (vx + F_e(t_0, t_1) : at_0 + bt_1 : ct_0 + d't_1),$$

where the coefficients a, b, c, d' and the binary form F_e satisfy conditions of the theorem. But $v = 1$, since A_5 is a simple group, and all its characters $A_5 \rightarrow \mathbb{C}^*$ are trivial. Obviously, we get (3.11).

Finally, we need to prove that all possibilities for G in (3.11) occur. It's sufficient to construct nonsingular curves \bar{C} invariant under an action of G_B with odd and even parameter d . We assume that G_B is a group consisting of maps

$$(x : t_0 : t_1) \mapsto (x : at_0 + bt_1 : ct_0 + d't_1),$$

with condition (3.10). Let d is even. Then the curve \bar{C} is represented by the following equation in $\mathbb{P}(4, 1, 1)$:

$$\Phi_3(t_0, t_1)x^2 + \Phi_2(t_0, t_1) = 0,$$

where $\Phi_i, i = 2, 3$ are binary forms in (3.4).

Let's construct the curve \bar{C} with odd d . Consider the equation of \bar{C} in $\mathbb{P}(30, 1, 1)$:

$$\Phi_1(t_0, t_1)x^2 + 2h(t_0, t_1)x + h'(t_0, t_1) = 0,$$

where Φ_1 is a binary form in (3.4), and h, h' are some binary forms in $\mathcal{I}^{St(\bar{A}_5)}$. It's easy to check by counting of parameters that h and h' can be chosen, such that $\text{Disc}(\bar{C}) = \Phi_1 h' - h^2$ has no multiple factors. Then \bar{C} is nonsingular. \square

3.3. Case, when $G_0 \simeq 1$, $G_K \simeq \mathbb{Z}_2$, and $\Sigma' \neq \Sigma$. Here we will apply arguments of [13, Section 3.2]. Let $r = |\Sigma|$, and $m = |\Sigma'|$.

Let $g_1 : \tilde{S} \rightarrow S$ be blowing up of the singular points of reducible fibres $\Sigma \setminus \Sigma'$, and $g_2 : \tilde{S} \rightarrow S'$ be the contraction of proper transform of $\Sigma \setminus \Sigma'$. The surface S' has $2(r - m)$ singular points of type A_1 . Obviously that maps g_1 and g_2 are G -equivariant. We have the G -equivariant commutative diagram.

$$(3.12) \quad \begin{array}{ccccc} & & \tilde{S} & & \\ & g_1 \swarrow & \downarrow \tilde{h} & \searrow g_2 & \\ S & & & & S' \\ \downarrow h & & & & \downarrow h' \\ & & \tilde{S}/\iota & & \\ & g'_1 \swarrow & & \searrow g'_2 & \\ S/\iota & & & & S'/\iota \end{array}$$

In the diagram the vertical arrows correspond to the quotient map by the involution ι , and maps g'_1 and g'_2 are induced by maps g_1 and g_2 . The triple (S, G, ϕ) defines a triple (S', G, ϕ') , where the morphism $\phi' : S' \rightarrow \mathbb{P}^1$ is induced by the morphism ϕ .

By [13, Lemma 3.4] we get that surfaces \tilde{S}/ι and S'/ι are nonsingular. Moreover, $S'/\iota \simeq \mathbb{F}_e$.

The morphism $h' : S' \rightarrow \mathbb{F}_e$ is a double cover branched over the union of curves $C' \cup g_{2*}(g_1^*(\Sigma \setminus \Sigma'))$, where the curve C' is the proper transform of curve C . The image of the curve $g_{2*}(g_1^*(\Sigma \setminus \Sigma'))$ on the ruled surface \mathbb{F}_e is the union of $r - m$ fibres. Denote these fibres as $S_i, i = 1, \dots, r - m$. Also let $\hat{C} = h'(C')$.

For each fiber S_i , $i = 1, \dots, r - m$ denote by x_{i1} and x_{i2} two distinct points of the intersection $S_i \cap \hat{C}$. Obviously, there is defined a faithful action of G_B (see exact sequence (3.2)) on \mathbb{F}_e . By [13, Lemma 3.5] the triple (S, G, ϕ) is minimal, iff points x_{i1} and x_{i2} lie in the same orbit under the action of G_B for each $i = 1, \dots, r - m$.

We consider cases $e = 0$ and $e > 0$ in Theorems 3.11 and 3.12 respectively.

We make some preparations before statement of Theorem 3.11. Introduce the coordinates $(x_0 : x_1, t_0 : t_1)$ on $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The morphism $\phi' : S' \rightarrow \mathbb{P}^1$ induces projection $\sigma : (x_0 : x_1, t_0 : t_1) \mapsto (t_0 : t_1)$. The fibres $S_i \subset \mathbb{F}_0$, $i = 1, \dots, r - m$ are represented by the equations:

$$S_i : a_i t_0 + b_i t_1 = 0.$$

Consider the form $Q_{r-m} = \prod_{i=1}^{r-m} (a_i t_0 + b_i t_1)$. We conjugate the subgroup $G_B \simeq A_5 \subset PGL(2, \mathbb{C})$ to the group $St(A_5)$. Let $\text{Stab}_{St(A_5)}(\sigma(S_i))$, $i = 1, \dots, r - m$ be a stabilizer of point $\sigma(S_i) \subset \mathbb{P}^1$ in the group $St(A_5)$. Considering equations (3.4), we get that $|\text{Stab}_{St(A_5)}(\sigma(S_i))|$ is either 1, or 2, or 3, or 5. Applying [13, Lemma 3.5], we get $\text{Stab}_{St(A_5)}(\sigma(S_i)) \simeq \mathbb{Z}_2$, $i = 1, \dots, r - m$. Hence $r - m = 30$, and $Q_{r-m} = \Phi_1$. The curve \hat{C} is represented by the equation:

$$(3.13) \quad \text{Equat}(\hat{C}) = p_0(t_0, t_1)x_0^2 + 2p_1(t_0, t_1)x_0x_1 + p_2(t_0, t_1)x_1^2 = 0,$$

where p_i , $i = 0, 1, 2$ are binary forms with degree $2d$. Note that the degree is even, so as the divisor class $\hat{C} + \sum_i S_i \in 2 \text{Pic } F_0$ (see [13, Lemma 3.6]), and $r - m = 30$ is even. The form $\text{Disc}(\hat{C}) = p_0 p_2 - p_1^2$ has no multiple factors, since \hat{C} is nonsingular. We will apply the Segre embedding $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ to represent the surface S by equations. Introduce the coordinates $(x : y : z : w)$ on \mathbb{P}^3 . This embedding is given by $\nu : (x_0 : x_1, t_0 : t_1) \mapsto (x_0 t_0 : x_0 t_1 : x_1 t_0 : x_1 t_1)$. We choose some polynomials $F_i(x, y, z, w)$, $i = 0, \dots, 2d + 28$, such that $x_0^i x_1^{2d+28-i} \Phi_1 \text{Equat}(\hat{C}) = \nu^*(F_i)$. The surface S is represented by the equations in $\mathbb{P}(d^{d+15}, 1^4)$ with the coordinates u_i , x , y , z , w , $i = 0, \dots, d + 14$:

$$(3.14) \quad \begin{aligned} u_i u_j &= F_{i+j}, \quad 0 \leq i \leq j \leq d + 14, \\ x^{j-i} u_i &= u_j z^{j-i}, \quad y^{j-i} u_i = u_j w^{j-i}, \quad 0 \leq i < j \leq d + 14, \\ xw &= yz. \end{aligned}$$

Theorem 3.11. *Let $(S, G, \phi) \in \mathbb{CB}$, and G be a finite nonsolvable group. Suppose that $G_0 \simeq 1$, $G_K \simeq \langle \iota \rangle \simeq \mathbb{Z}_2$, $\Sigma' \neq \Sigma$ and $S'/\iota \simeq \mathbb{F}_0$. Then there is defined a faithful action of G_B (see (3.2)) on \mathbb{F}_0 , and $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. There is a G -invariant birational map $(S, G, \phi) \dashrightarrow (S', G, \phi')$ described in the diagram (3.12). The surface S' can be represented by equations (3.14). The parameter d in (3.13) is odd. The morphism $\phi' : S' \rightarrow \mathbb{P}^1$ is given by*

$$\phi' : (u_0 : \dots : u_{d+14} : x : y : z : w) \mapsto \begin{cases} (x : y), & \text{if } (x : y) \neq (0 : 0); \\ (z : w), & \text{if } (z : w) \neq (0 : 0). \end{cases}$$

We have

$$(3.15) \quad \Phi_1(t_0, t_1) \nmid (p_0(t_0, t_1)t_0^2 + 2p_1(t_0, t_1)t_0t_1 + p_2(t_0, t_1)t_1^2),$$

where p_i , $i = 0, 1, 2$ are binary forms in (3.13), and Φ_1 is the binary form from (3.4).

The group $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ is the image of diagonal embedding $St(A_5) \hookrightarrow PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. The group G is isomorphic to \bar{A}_5 . This possibility for G occur.

The group G acts on S' by the following way. Embedding of $G_B \simeq St(A_5)$ to $PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ defines a unique embedding $St(\bar{A}_5) \hookrightarrow SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. This defines an action of $St(\bar{A}_5)$ on the surface S' given by the equation (3.14). The action of $St(\bar{A}_5)$ on coordinates u_i , $i = 0, \dots, d+14$ coincides with the action on monomials $x_0^i x_1^{d+14-i}$, $i = 0, \dots, d+14$. An action of G is generated by the action of $St(\bar{A}_5)$ and by the map

$$(u_0 : \dots : u_{d+14} : x : y : z : w) \rightarrow (-u_0 : \dots : -u_{d+14} : x : y : z : w).$$

Proof. The subgroup $G_B \subset \text{Aut}(\mathbb{F}_0)$ acts trivially on $\text{Pic}(\mathbb{F}_0)$. Hence $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. Consider projections $\pi_i : PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C}) \rightarrow PGL(2, \mathbb{C})$, $i = 1, 2$ on the first and the second factor respectively. By Goursat's Lemma (see [4, Lemma 4.1]) we get that $G_B \simeq \pi_1(G_B) \Delta_D \pi_2(G_B)$, where $D \simeq \text{Im}(\pi_1|_{G_B}) / \text{Ker}(\pi_2|_{G_B})$. We have $D \simeq 1$ or A_5 .

By [13, Lemma 3.5] we need to find conditions, when points x_{i1} and x_{i2} for each $i = 1, \dots, 30$ lie in the same orbit under an action of G_B . Obviously, $D \simeq A_5$. We conjugate $G_B \subset PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ to the image of diagonal embedding $St(A_5) \hookrightarrow PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$. It's easy to check that none of points x_{i1} and x_{i2} for each $i = 1, \dots, 30$ lies on the diagonal $x_0 t_1 - x_1 t_0 = 0$. This is a sufficient condition, and it's equivalent to (3.15).

Therefore we need to prove existence of curves $\hat{C} \subset \mathbb{F}_0$ with odd parameter d , such that condition (3.15) holds. Also we will prove that d cannot be even. We will use notations and arguments of Lemma 3.9. Consider a linear system \mathcal{J} of $St(\bar{A}_5)$ -invariant curves with bidegree $(2, 2d)$ in \mathbb{F}_0 . Consider the projection $\xi : R_2 \otimes R_{2d} \rightarrow R_{2d+2}$. It's given by the polynomial $p_0 t_0^2 + 2p_1 t_0 t_1 + p_2 t_1^2$.

Suppose that d is even. It's easy to check that any polynomial $f(t_0, t_1) \in R_{2d+2}^{St(\bar{A}_5)}$ is divided by Φ_1 . Therefore it's impossible.

Let d is odd. In Lemma 3.9 we proved that a general member of \mathcal{J} is nonsingular, if $d = 30k + 15$, $k \in \mathbb{N}$, $k \geq 2$. But it's obvious that there exist polynomial $f(t_0, t_1) \in R_{2d+2}^{St(\bar{A}_5)}$ with degree $2d + 2 = 60k + 32$, $k \geq 2$, which is not divided by Φ_1 .

The remaining arguments follow from the construction of equations (3.14) and are obvious. \square

In the next theorem we consider case $e > 0$.

Theorem 3.12. *Let $(S, G, \phi) \in \mathbb{CB}$, and G be a finite nonsolvable group. Suppose that $G_0 \simeq 1$, $G_K \simeq \langle \iota \rangle \simeq \mathbb{Z}_2$, $\Sigma' \neq \Sigma$, and $S'/\iota \simeq \mathbb{F}_e$, $e > 0$. Then there is a G -invariant birational map $(S, G, \phi) \dashrightarrow (S', G, \phi')$ described in diagram (3.12). The surface S' contains a G -invariant curve E , which is the preimage of exceptional section \mathbb{F}_e . Consider the contraction of this curve $(S', G, \phi') \rightarrow (S'', G, \phi'')$, where a map $\phi'' : S'' \dashrightarrow \mathbb{P}^1$ is defined by ϕ' . The surface S'' is given by the following equation in $\mathbb{P}(d + e + 15, e, 1, 1)$ with the coordinates $(u : x : t_0 : t_1)$:*

$$(3.16) \quad u^2 + \Phi_1(t_0, t_1)(p_0(t_0, t_1)x^2 + 2p_1(t_0, t_1)x + p_2(t_0, t_1)) = 0,$$

where p_i , $i = 0, 1, 2$ are binary forms with degree $2d$, $2d + e$, $2d + 2e$, respectively, and Φ_1 is the binary form from (3.4). Also $\Phi_1 \nmid (p_0 p_2 - p_1^2)$.

The map ϕ'' is given by

$$\phi'' : (u : x : t_0 : t_1) \mapsto (t_0 : t_1).$$

Moreover, $e \equiv 2 \pmod{4}$. The group G is generated by the maps:

$$u \mapsto -u,$$

$$(u : x : t_0 : t_1) \mapsto (u : x + F_e(t_0, t_1) : at_0 + bt_1 : ct_0 + d't_1),$$

where

$$(3.17) \quad \begin{pmatrix} a & b \\ c & d' \end{pmatrix} \in St(\bar{A}_5),$$

and $F_e(t_0, t_1)$ is a some binary form with degree e , unique for each matrix (3.17). The group G is isomorphic to

$$(3.18) \quad G \simeq \begin{cases} \bar{A}_5, & \text{if } d \text{ is even,} \\ \mathbb{Z}_2 \times A_5, & \text{if } d \text{ is odd.} \end{cases}$$

All possibilities for G occur.

Proof. We will use the following construction to represent the surface S'' by equations. Consider the morphism $\mathbb{F}_e \rightarrow \mathbb{P}(e, 1, 1)$, which is the blowing down of exceptional section \mathbb{F}_e . Introduce on $\mathbb{P}(e, 1, 1)$ the coordinates $(x : t_0 : t_1)$. The map $\phi' : S' \rightarrow \mathbb{P}^1$ induces the projection $\sigma : (x : t_0 : t_1) \mapsto (t_0 : t_1)$. The fibres $S_i \subset \mathbb{F}_e$, $i = 1, \dots, r - m$ are represented by the equations:

$$S_i : a_i t_0 + b_i t_1 = 0.$$

Consider the form $Q_{r-m} = \prod_{i=1}^{r-m} (a_i t_0 + b_i t_1)$. We conjugate the group $G_B \simeq A_5 \subset PGL(2, \mathbb{C})$ to the group $St(A_5)$. Let $\text{Stab}_{St(A_5)}(\sigma(S_i))$, $i = 1, \dots, r - m$ be a stabilizer of point $\sigma(S_i) \subset \mathbb{P}^1$ in the group $St(A_5)$. Considering equations (3.4), we get that $|\text{Stab}_{St(A_5)}(\sigma(S_i))|$ is either 1, or 2, or 3, or 5. Applying [13, Lemma 3.5], we get $\text{Stab}_{St(A_5)}(\sigma(S_i)) \simeq \mathbb{Z}_2$, $i = 1, \dots, r - m$. Hence $r - m = 30$, and $Q_{r-m} = \Phi_1$ (see (3.4)). The curve \hat{C} is represented by the following equation in $\mathbb{P}(e, 1, 1)$ with the coordinates $(x : t_0 : t_1)$:

$$p_0(t_0, t_1)x^2 + 2p_1(t_0, t_1)x + p_2(t_0, t_1) = 0.$$

Each fibre S_i , $i = 1, \dots, 30$ intersects the curve \hat{C} in two distinct points: x_{i1} and x_{i2} . Hence $\Phi_1 \nmid (p_0 p_2 - p_1^2)$. We construct a double cover of $\mathbb{P}(e, 1, 1)$ branched along \hat{C} and $\Phi_1(t_0, t_1) = 0$. We get the surface S'' given by the equation (3.16).

Note that $\deg(p_0)$ is even, since $\hat{C} + \sum_i S_i \in 2\text{Pic}(\mathbb{F}_e)$ (see [13, Lemma 3.6]), and $r - m = 30$ is even. Denote the degree as $2d$.

Then we apply the arguments as in Theorem 3.10. We prove that e is even, and conjugate G_B to a group consisting of the following maps

$$(x : t_0 : t_1) \mapsto (x + F_e(t_0, t_1) : at_0 + bt_1 : ct_0 + d't_1),$$

where the coefficients a, b, c, d' and the binary form F_e satisfy conditions of the Theorem.

By [13, Lemma 3.5] we need to find conditions, when the points x_{i1} and x_{i2} for each $i = 1, \dots, 30$ lie in the same orbit under an action of G_B . Obviously, it's sufficient to check, that each element $g \in G_B$ with $\text{ord}(g) = 2$ doesn't have fixed points on the curve \hat{C} . The element g can be conjugated to the map:

$$(x : t_0 : t_1) \mapsto (x : it_0 : -it_1).$$

It's easy to see that g doesn't have fixed points on \hat{C} , iff $e \equiv 2 \pmod{4}$. We easily get (3.18).

We need to prove existence of curves $\hat{C} \subset \mathbb{F}_e$ with odd and even parameters d , such that listed above conditions holds. We assume that G_B is a group consisting of maps

$$(x : t_0 : t_1) \mapsto (x : at_0 + bt_1 : ct_0 + d't_1),$$

with condition (3.17).

Let d is even. We take $e = 34$, and the curve \hat{C} is represented by the following equation in $\mathbb{P}(34, 1, 1)$ with the coordinates $(x : t_0 : t_1)$

$$\Phi_3(t_0, t_1)x^2 + Q_{60}(t_0, t_1)\Phi_2(t_0, t_1) = 0,$$

where Φ_2 and Φ_3 are binary forms from (3.4), and $Q_{60} \in \mathcal{I}^{St(\bar{A}_5)}$ (see Notation 3.6) is a some binary form with degree 60 and without multiple factors.

Let d is odd. We can employ here example constructed in proof of Theorem 3.10. \square

3.4. Case, when $G_0 \simeq 1$, $G_K \simeq \mathbb{Z}_2^2$. In this section we prove the next theorem.

Theorem 3.13. *Let $(S, G, \phi) \in \mathbb{CB}$, and G be a finite nonsolvable group. Suppose that $G_0 \simeq 1$, $G_K \simeq \mathbb{Z}_2^2$. Then there exists an embedding $S \hookrightarrow \mathbb{P}(\mathcal{E})$, where \mathcal{E} is a line bundle on \mathbb{P}^1 . We have $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$, and isomorphisms $f_i : \mathcal{E}_i \rightarrow \mathcal{O}(a_i)$, $i = 0, 1, 2$, $a_0 = 0$, $0 \leq a_1 \leq a_2$. The surface S can be represented by the equation in $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$:*

$$(3.19) \quad \sum_{i=0}^2 \sum_{j,k=0}^{a_i} p_i^{j,k}(t_0, t_1) \xi_i^j \xi_i^k = 0,$$

where $p_i^{j,k}$ are binary forms with degree d , and $\xi_i^j = f_i^{-1}(t_0^j t_1^{a_i-j})$, $i = 0, 1, 2$, $0 \leq j \leq a_i$. The morphism $\phi : S \rightarrow \mathbb{P}^1$ is induced by the natural projection $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$. The following conditions holds.

$$(3.20) \quad \begin{aligned} H_0 &= p_0^{0,0}(t_0, t_1) \in \mathcal{I}^{St(\bar{A}_5)}, \\ H_1 &= \sum_{j,k=0}^{a_1} p_1^{j,k}(t_0, t_1) t_0^{j+k} t_1^{2a_1-j-k} \in \mathcal{I}^{St(\bar{A}_5)}, \\ H_2 &= \sum_{j,k=0}^{a_2} p_2^{j,k}(t_0, t_1) t_0^{j+k} t_1^{2a_2-j-k} \in \mathcal{I}^{St(\bar{A}_5)}. \end{aligned}$$

Also the binary forms H_0 , H_1 and H_2 do not have multiple and pairwise common factors.

The group G_K acts by the following way (see Theorem 3.4).

$$\begin{aligned} \iota_0(\xi) &= \mp \xi_0 \pm \xi_1 \pm \xi_2, \\ \iota_1(\xi) &= \pm \xi_0 \mp \xi_1 \pm \xi_2, \\ \iota_2(\xi) &= \pm \xi_0 \pm \xi_1 \mp \xi_2, \end{aligned}$$

for any $\xi = \xi_0 + \xi_1 + \xi_2$, $\xi_i \in \mathcal{E}_i$, $i = 0, 1, 2$.

The action of G on the surface S is generated by the action of G_K and an action of $St(\bar{A}_5)$. The action of $St(\bar{A}_5)$ on sections $\xi_i^j = f_i^{-1}(t_0^j t_1^{a_i-j})$, $i = 0, 1, 2$,

$0 \leq j \leq a_i$ is induced by the action on $\mathbb{C}[t_0, t_1]$. We have

$$(3.21) \quad G \simeq \begin{cases} \mathbb{Z}_2 \times \bar{A}_5, & \text{if either } a_1, \text{ or } a_2 \text{ is odd,} \\ \mathbb{Z}_2^2 \times A_5, & \text{otherwise.} \end{cases}$$

All possibilities for G occur.

Proof. Denote as f the fibre's divisor class of the conic bundle (S, G, ϕ) . We have $-K_S \cdot f = 2$. It's well known that a line bundle $\mathcal{O}(-K_S)$ is locally free of rank 3. Hence the line bundle $\mathcal{O}(-K_S)$ is relatively very ample and defines an embedding $S \hookrightarrow \mathbb{P}(\mathcal{E}')$, where $\mathcal{E}' = \phi_*(\mathcal{O}(-K_S))$. By Grothendieck theorem we have $\mathcal{E}' = \mathcal{O}(a'_0) \oplus \mathcal{O}(a'_1) \oplus \mathcal{O}(a'_2)$. Obviously, we can take $a'_0 \leq a'_1 \leq a'_2$. Hence we can take the bundle \mathcal{E} in the statement of theorem to be equal $\mathcal{E} = \mathcal{E}' \otimes \mathcal{O}(-a'_0)$.

An action of G on $\mathcal{O}(-K_S)$ defines an action on \mathcal{E} . In the next lemma we show that the action of G_K on \mathcal{E} is diagonalizable.

Lemma 3.14. *We can choose a decomposition $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_i \simeq \mathcal{O}(a_i)$, $i = 0, 1, 2$, such that G_K acts by the following way. Denote three different nontrivial elements in $G_K \simeq \mathbb{Z}_2^2$ as $\iota_0, \iota_1, \iota_2$. Then*

$$(3.22) \quad \begin{aligned} \iota_0(\xi) &= \mp \xi_0 \pm \xi_1 \pm \xi_2, \\ \iota_1(\xi) &= \pm \xi_0 \mp \xi_1 \pm \xi_2, \\ \iota_2(\xi) &= \pm \xi_0 \pm \xi_1 \mp \xi_2, \end{aligned}$$

for any $\xi = \xi_0 + \xi_1 + \xi_2$, $\xi_i \in \mathcal{E}_i$, $i = 0, 1, 2$.

Proof. Recall that $a_0 = 0$, $0 \leq a_1 \leq a_2$. Also remind that G_K acts trivially on the base of fibration ϕ .

Suppose that $0 < a_1 < a_2$. Then each bundle \mathcal{E}_i , $i = 0, 1, 2$ is invariant under the action of G_K . Hence the statement is obvious.

Suppose that $a_i = a_j$, $a_i \neq a_k$ for some $i \neq j \neq k \neq i$. Without loss of generality we can take $0 = a_1 < a_2$. Then the action of G_K on $\mathcal{E}_0 \oplus \mathcal{E}_1$ defines an embedding $G_K \hookrightarrow GL(2, \mathbb{C})$. But, obviously, any subgroup $\mathbb{Z}_2^2 \subset GL(2, \mathbb{C})$ is diagonalizable.

Suppose that $0 = a_1 = a_2$. Then the statement follows from the fact that any subgroup $\mathbb{Z}_2^2 \subset GL(3, \mathbb{C})$ is diagonalizable. \square

We apply Lemma 3.14. Fix isomorphisms $f_i : \mathcal{E}_i \rightarrow \mathcal{O}(a_i)$, $i = 0, 1, 2$. Let $\xi_i^j = f_i^{-1}(t_0^j t_1^{a_i-j})$, $i = 0, 1, 2$, $0 \leq j \leq a_i$ be generators of global section spaces of bundles \mathcal{E}_i . Then the surface S is presented by the following equation in $\mathbb{P}(\mathcal{E})$

$$\sum_{0 \leq i \leq j \leq 2} \sum_{k \leq a_i, l \leq a_j} p_{i,j}^{k,l}(t_0, t_1) \xi_i^k \xi_j^l = 0,$$

where $p_{i,j}^{k,l}(t_0, t_1)$ are binary forms with degree d . But it easily follows from equations (3.22) that $p_{i,j}^{k,l} = 0$, if $i \neq j$. Hence the surface S can be represented by equation (3.19).

Let's find relations on the forms p_i^j , $0 \leq j \leq a_i$, $i = 0, 1, 2$. By Theorem 3.4 each nontrivial element ι_i , $i = 0, 1, 2$ of the subgroup G_K fixes pointwise an irreducible smooth bisection C_i . Hence, there is defined an action of G on the set of these curves, since $G_K \triangleleft G$. This action defines a homomorphism $\sigma : G \rightarrow S_3$. But $G_B \simeq A_5$ is simple. Therefore σ is trivial. The curves C_i , $i = 0, 1, 2$ on the surface S are cut out by the hypersurfaces:

$$\xi_i^j = 0, \quad 0 \leq j \leq a_i.$$

We conjugate $G_B \subset PGL(2, \mathbb{C})$ to $St(A_5)$. We employ now notations (3.20). From triviality of σ we get: $H_0H_1, H_0H_2, H_1H_2 \in \mathcal{I}^{St(\bar{A}_5)}$. Hence $H_0^2, H_1^2, H_2^2 \in \mathcal{I}^{St(\bar{A}_5)}$. One knows that all characters of $St(\bar{A}_5)$ are trivial (see (3.4)). Therefore we get (3.20).

It's easy to check that the surface S is nonsingular, iff the binary forms H_0, H_1 and H_2 do not have multiple and pairwise common factors.

Now we can describe an action of group G on the surface S . The action of $St(\bar{A}_5)$ on $\mathbb{C}[t_0, t_1]$ induces an action on $\xi_i^j = f_i^{-1}(t_0^j t_1^{a_i-j})$, $i = 0, 1, 2$, $0 \leq j \leq a_i$. The action of G is generated by the action of G_K and the action of $St(\bar{A}_5)$. Therefore we easily get (3.21). It's easy to check that all possibilities for G in (3.21) occur. \square

4. CONJUGACY QUESTION.

The main result of this section is Theorem 4.2. Here we reprove results of [4, Section 8] for the sake of completeness. As the main tool we will use the next theorem.

Theorem 4.1 ([10, Theorem 1.6]). (1) *Let (S, G) be a surface in the class \mathbb{D} with degree $K_S^2 = 1$, and let $\chi : S \dashrightarrow S'$ be a birational G -invariant map onto an arbitrary surface $(S', G) \in \mathbb{D} \cup \mathbb{CB}$. Then S' , like S , is a del Pezzo surface of degree 1 and χ is an isomorphism.*
(2) *Let $\chi : S \dashrightarrow S'$ be a birational map, where $(S, G) \in \mathbb{D}$ and $(S', G) \in \mathbb{D} \cup \mathbb{CB}$. Suppose that S has no points x with $|\text{Orb}_G(x)| < K_S^2$, where $\text{Orb}_G(x)$ is an orbit of point x under action of G . Then χ is an isomorphism.*
(3) *Let $\chi : S \dashrightarrow S'$ be a birational G -invariant map, where $(S, G, \phi) \in \mathbb{CB}$ and $(S', G) \in \mathbb{D} \cup \mathbb{CB}$. Suppose that $K_S^2 \leq 0$; then $(S', G, \phi') \in \mathbb{CB}$, $K_S^2 = K_{S'}^2$, and χ takes a pencil of conics on S to a pencil of conics on S' , that is, the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\chi} & S' \\ \phi \downarrow & & \downarrow \phi' \\ \mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^1 \end{array}$$

is commutative, where π is an isomorphism over \mathbb{C} .

Theorem 4.2. (1) *Let (S, G, ϕ) be a surface in the class \mathbb{CB} , $K_S^2 \leq 0$, and G be a finite nonsolvable group. Let $\chi : S \dashrightarrow S'$ be a birational G -invariant map, where $(S', G) \in \mathbb{D} \cup \mathbb{CB}$. Then $(S', G, \phi') \in \mathbb{CB}$, and $K_S^2 = K_{S'}^2$. The map χ is a composition of elementary transformations $\text{elm}_{x_1} \circ \dots \circ \text{elm}_{x_n}$, where (x_1, \dots, x_n) is a G -invariant set of points not lying on a singular fibre with no two points lying on the same fibre.*

(2) *Let (S, G, ϕ) be a surface in the class \mathbb{CB} , $K_S^2 > 0$, and G be a finite nonsolvable group. Let $\chi : S \dashrightarrow S'$ be a birational G -invariant map, where $(S', G) \in \mathbb{D} \cup \mathbb{CB}$. Then $K_S^2 = 8$, $S \simeq \mathbb{F}_n$, $n \neq 1$.*

(a) *Let n is odd. Then one the following cases occurs*

$$(4.1) \quad \begin{aligned} &S' \simeq \mathbb{P}^2, \text{ and } G \simeq \mathbb{Z}_{n'} \times \bar{A}_5; \\ &S' \simeq \mathbb{F}_m, \text{ where } m \text{ is odd and } m \neq 1. \end{aligned}$$

(b) *Let n is even. Then S' can be isomorphic to \mathbb{F}_m , where m is even. If $n = 0$, and $m \neq 0$, then $G \simeq \mathbb{Z}_m \times A_5$.*

- (3) Let (S, G) be a surface in the class \mathbb{D} , and G be a finite nonsolvable group. Let $\chi : S \dashrightarrow S'$ be a birational G -invariant map, where $(S', G) \in \mathbb{D} \cup \mathbb{CB}$. Then we have the following.
- (a) Let $S \simeq \mathbb{P}^2$, and $G \simeq \mathbb{Z}_m \times \bar{A}_5$. Then the surface S' may be isomorphic to either \mathbb{P}^2 , or \mathbb{F}_n , where n is odd.
 - (b) Otherwise we have $S' \simeq S$. If $K_S^2 < 9$, then χ is an automorphism of S .

Remark 4.3. I. Cheltsov proved the following. If $(\mathbb{F}_n, G, \phi) \in \mathbb{CB}$, $n \neq 1$, and G be a finite nonsolvable group, then there exist a birational G -invariant map $\chi : \mathbb{F}_n \dashrightarrow S'$, where

$$\begin{aligned} S' &\simeq \mathbb{P}^2, \text{ if } n \text{ is odd;} \\ S' &\simeq \mathbb{F}_0, \text{ if } n \text{ is even.} \end{aligned}$$

Proof. The first case of theorem follows from Theorem 4.1 and [4, Theorems 7.7, Proposition 7.14].

Let's prove the second case of theorem. First we prove that $K_S^2 = 8$. Denote by r the number of the reducible fibres of conic bundle (S, G, ϕ) . Suppose that $r \neq 0$. Obviously, we have $G_B \simeq A_5$ (see exact sequence (3.2) and Klein's classification of the finite subgroups in $PGL(2, \mathbb{C})$ in [4, Section 5.5]). Then by (3.4) we have $r \geq 12$. Hence by Noether formula $K_S^2 = 8 - r \leq -4$. Therefore $r = 0$, $K_S^2 = 8$.

We will use results of Theorem 3.1 and theory of elementary links (see [4, Section 7.2]). By [4, Theorem 7.7] the map χ is equal to a composition of elementary links $\chi_1 \circ \dots \circ \chi_k$. It's easy to check that χ_k is an elementary link of type II (see Theorem 3.1). We have $\chi_k(\mathbb{F}_n) \simeq \mathbb{F}_l$. Then we apply Theorem 3.1. For any point $x \in \mathbb{P}^1$ we have $|\text{Orb}_{St(A_5)}(x)|$ is even (see Notation 3.5). Hence we easily get that $l - n$ is even.

Therefore if n is even, then χ_i , $i = 1, \dots, k$ are elementary links of type II. And we easily get the case 2b of theorem.

Consider case, when n is odd. Then $G \simeq \mathbb{Z}_{n'} \times \bar{A}_5$ by Theorem 3.1. Suppose that χ is not a composition of elementary links of type II. Then one of elementary links χ_i , $1 \leq i \leq k-1$ must be a link of type III. We may suppose that the links χ_j , $i < j \leq k$ are of type II. In this case $\chi_i \circ \dots \circ \chi_k(S) \simeq \mathbb{P}^2$. Below we will see that $S' \not\simeq X$, where $(X, G) \in \mathbb{D}$, and $K_X^2 < 9$. Therefore we get the case 2a of theorem.

Let's prove the third case of theorem. To apply Theorem 4.1, we need to show, that S has no points x with $|\text{Orb}_G(x)| < K_S^2$. We will argue, considering different values of K_S^2 .

If K_S^2 is equal to either 7, or 6, or 4, or 1, then by Theorem 2.3 there is no such pairs (S, G) .

Let $K_S^2 = 2$. We apply Theorem 2.6. Consider the following two automorphisms of surface S , given by equation (2.4):

$$\begin{aligned} \alpha : (T_0 : T_1 : T_2 : T_3) &\mapsto (T_1 : T_2 : T_0 : T_3), \\ \beta : (T_0 : T_1 : T_2 : T_3) &\mapsto (\varepsilon_7 T_0 : \varepsilon_7^4 T_1 : \varepsilon_7^2 T_2 : T_3). \end{aligned}$$

It's easy to check by calculations that $\alpha, \beta \in G$, and there is no point $x \in S$ fixed under an action of subgroup $H \subset G$ generated by α and β .

Let $K_S^2 = 3$. We apply Theorem 2.5. Consider the maps (2.3). Again by easy calculations we can check that there is no point $x \in S$, such that $|\text{Orb}_G(x)| < 3$.

Let $K_S^2 = 5$. We apply Theorem 2.4. It's sufficient to consider the case $G \simeq A_5$. Suppose that there is a point $x \in S$, such that $|\text{Orb}_G(x)| < K_S^2 = 5$. Denote by

$\text{Stab}_G(x)$ the stabilizer of point x in the group G . Then $\text{Stab}_G(x)$ is a subgroup of G with order either 15, or 20, or 30 or 60. It's well known that there are no subgroups of A_5 with order either 15, or 20, or 30. Hence $|\text{Stab}_G(x)| = 60 = |G|$. But it's easy to see from (2.2) that the action of group G on the surface S has no fixed points. Contradiction.

Let $K_S^2 = 8$. We apply Theorem 2.2. Consider the action of $St(A_5)$ on \mathbb{P}^1 . It's known (see (3.4)) that $|\text{Orb}_{St(A_5)}(x)| \geq 12$ for any point $x \in \mathbb{P}^1$. Therefore $|\text{Orb}_G(x)| \geq 12$ for any point $x \in S$.

The case $K_S^2 = 9$ easily follows from the above investigation and Theorem 2.1. However in this case the condition $|\text{Orb}_{St(A_5)}(x)| \geq 9$ for any point $x \in S \simeq \mathbb{P}^2$ not always holds. For example in the second case of Theorem 2.1. Therefore we cannot apply Theorem 4.1 to prove that χ is an isomorphism. But in fact we need only to know an isomorphism class of S' to describe conjugacy classes. \square

Corollary 4.4. *Suppose that the conditions of case 1 of Theorem 4.2 are satisfied. After suppose that $G_K \not\cong \mathbb{Z}_2$ (see (3.2)). Then χ is an isomorphism.*

Proof. By Theorems 3.4 and 3.7 we have $G_K \simeq \mathbb{Z}_2^2$ or D_n , $n \geq 2$. Therefore it's easy to see that any G -orbit of points (x_1, \dots, x_n) has two points lying on the same fibre of conic bundle ϕ . Contradiction to case 1 of Theorem 4.2. \square

5. THE LIST OF FINITE NONSOLVABLE SUBGROUPS IN $\text{Cr}_2(\mathbb{C})$.

Summarizing results obtained in sections 2 and 3 we get the following list of finite nonsolvable subgroups in $\text{Cr}_2(\mathbb{C})$.

- $L_2(7)$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, L_2(7)) \in \mathbb{D}$, which were described in Theorems 2.1 and 2.6.
- $\mathbb{Z}_2 \times L_2(7)$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, \mathbb{Z}_2 \times L_2(7)) \in \mathbb{D}$, which were described in Theorem 2.6.
- A_6
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, A_6) \in \mathbb{D}$, which were described in Theorem 2.1.
- S_5
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, S_5) \in \mathbb{D}$, which were described in Theorems 2.4, 2.5.
- $St(A_5) \wr \langle \tau \rangle$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, St(A_5) \wr \langle \tau \rangle) \in \mathbb{D}$, which were described in Theorem 2.2.
- $A_5 \times A_5, A_5 \times S_4, A_5 \times A_4$.
These groups are presented in $\text{Cr}_2(\mathbb{C})$ by triples $(S, G, \phi) \in \mathbb{CB}$ (G is one of our groups), which were described in Theorem 3.1.
- $D_n \times \bar{A}_5, n \geq 3$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by triples $(S, D_n \times \bar{A}_5, \phi) \in \mathbb{CB}$, which were described in Theorem 3.7.
- $D_n \times A_5, n \geq 3$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by triples $(S, D_n \times A_5, \phi) \in \mathbb{CB}$, which were described in Theorems 3.1 and 3.7.

- $\mathbb{Z}_n \times \bar{A}_5$, $n \geq 3$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, \mathbb{Z}_n \times \bar{A}_5) \in \mathbb{D}$ and triples $(S, \mathbb{Z}_n \times \bar{A}_5, \phi)$, which were described in Theorems 2.1, 3.1.
- $\mathbb{Z}_n \times A_5$, $n \geq 3$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by triples $(S, \mathbb{Z}_n \times A_5, \phi) \in \mathbb{CB}$, which were described in Theorem 3.1.
- $\mathbb{Z}_2^2 \times \bar{A}_5$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by triples $(S, \mathbb{Z}_2^2 \times \bar{A}_5, \phi) \in \mathbb{CB}$, which were described in Theorem 3.7.
- $\mathbb{Z}_2^2 \times A_5$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by triples $(S, \mathbb{Z}_2^2 \times A_5, \phi) \in \mathbb{CB}$, which were described in Theorems 3.1, 3.7 and 3.13.
- $\mathbb{Z}_2 \times \bar{A}_5$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, \mathbb{Z}_2 \times \bar{A}_5) \in \mathbb{D}$ and triples $(S, \mathbb{Z}_2 \times \bar{A}_5, \phi) \in \mathbb{CB}$, which were described in Theorems 2.1, 3.1, and 3.13.
- $\mathbb{Z}_2 \times A_5$
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, \mathbb{Z}_2 \times A_5) \in \mathbb{D}$ and triples $(S, \mathbb{Z}_2 \times A_5, \phi) \in \mathbb{CB}$, which were described in Theorems 2.2, 3.1, 3.8, 3.10, 3.12.
- \bar{A}_5
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, \bar{A}_5) \in \mathbb{D}$ and triples $(S, \bar{A}_5, \phi) \in \mathbb{CB}$, which were described in Theorems 2.1, 3.1, 3.8, 3.10, 3.11, 3.12.
- A_5
This group is presented in $\text{Cr}_2(\mathbb{C})$ by pairs $(S, A_5) \in \mathbb{D}$ and triples $(S, A_5, \phi) \in \mathbb{CB}$, which were described in Theorems 2.1, 2.4, 3.1.

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